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The isoperimetric Deficit upper limit for the convex body in R^n

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Abstract. In this paper, we investigate the isoperimetric deficit upper limit for a convex body in the Euclidean space R^n. Some isoperimetric deficit upper limits are obtained. These limits, obtained in this paper, are fundamental generalizations of the known Bottema's result for oval domain in the Euclidean plane R^2.

1 Introductions and Preliminaries

The problems, investigated in integral geometry, has fundamental background and applications to stochastic geometry, stereology, metal science, mineralogic science, tomography imagine science, material science, biological science, medical science, information science and etc..

The sets of random geometric subjects (lines, planes, linear spaces, compact submanifolds, convex bodies and etc.) are fundamental in integral geometry. It is natural to introduce the invariant geometric measure for those sets. We are also interested in the relation among invariants of a geometric subject, such as, volume,

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surface area and the curvature of the surface of the subject investigated. Usually, the relations among those invariants are some equalities and inequalities and are, respectively, called geometric equalities and geometric inequalities.

A set of points K in the Euclidean space R^n is convex if for all $x, y \in K$ and $0 < \lambda < 1$, $\lambda x + (1-\lambda)y \in K$. A convex body is a compact convex set with nonempty interiors. The convex hull K^* of a set of points K in R^n is the intersection of all convex bodies that contain K. The **Minkowski sum** and **scalar product** of convex sets K and L for $\lambda \geq 0$ are, respectively, defined by $K + L = \{x + y : x \in K,\ y \in L\}$ and $\lambda K = \{\lambda x : \quad x \in K\}$. A **homothety** of a convex set K is of the form $x + \lambda K$ for $x \in R^n$, $\lambda > 0$ (a translation and a dilation).

The **support function** of a convex body K is defined by

$$h(K, x) = \sup\{x \cdot y : \quad y \in K\}, \quad x \in R^n,$$

and the width function of $K$is defined as

$$W(K, u) = h(K, u) + h(K, -u), \quad u \in S^{n-1}.$$

The width of K is defined by

$$W = \inf_{u \in S^{n-1}} \{W(K, u)\}.$$

Perhaps the oldest geometric inequality is the well-known isoperimetric inequality. It says that the ball encloses the maximum volume among all domains of fixed surface area in the Euclidean space R^n:

Proposition 1. *Let K be a domain in the Euclidean space R^n, then the volume V and the surface area A of K satisfies*

$$A^n - n^n \omega_n V^{n-1} \geq 0. \tag{1.1}$$

The equality sign holds if and only if K is a ball.

Here ω_n is the volume of the unit ball and its value is:

$$\omega_n = \frac{2\pi^{n/2}}{n\Gamma\left(\frac{n}{2}\right)}, \tag{1.2}$$

where Γ is the Gamma function.

The isoperimetric deficit

$$\Delta_n(K) = A^n - n^n \omega_n V^{n-1} \tag{1.3}$$

measures the difference between a domain K of surface area A and volume V, and a ball of radius $\left(\frac{A}{n\omega_n}\right)^{\frac{1}{n-1}}$ (where the index n is the dimension of the space R^n).

Mathematicians would be interested in the inequalities of the form (called the Bonnesen type inequalities):

$$\Delta_n(K) = A^n - n^n\omega_n V^{n-1} \geq B_K, \tag{1.4}$$

where the quantity B_K is an invariant of geometric significance having the following basic properties:

1. B_K is non-negative;
2. B_K is vanish only when K is a ball.

Since for any domain K in R^2, its convex hull K^* increases the area A^* and decreases the perimeter P^*. Then we have $P^2 - 4\pi A \geq P^{*2} - 4\pi A^*$, that is, $\Delta_2(K) \geq \Delta_2(K^*)$. Therefore the isoperimetric inequality and the Bonnesen-type inequality are valid for all domains in R^2 if these inequalities are valid for convex domains. But for a domain in space R^n, the convex hull does not always increase the volume and at the same time decrease the surface area. Therefore the convexity of domain is fundamental for isoperimetric problem in space R^n.

The isoperimetric inequality (1.1) can be equivalently rewritten as

$$\frac{nV}{A} \leq \sqrt[n]{\frac{V}{\omega_n}} \leq \sqrt[n-1]{\frac{A}{n\omega_n}}. \tag{1.5}$$

Then a Bonnesen-style inequality

$$A^n - n^n\omega_n V^{n-1} \geq B_K \tag{1.6}$$

may be the form of

$$\left(\frac{A}{n\omega_n}\right)^n - \left(\frac{V}{\omega_n}\right)^{n-1} \geq B'_K, \tag{1.7}$$

or

$$\left(\frac{A}{n\omega_n}\right)^{\frac{n}{n-1}} - \frac{V}{\omega_n} \geq B''_K, \tag{1.8}$$

or

$$\left(\frac{A}{n\omega_n}\right) - \left(\frac{nV}{A}\right)^{n-1} \geq B'''_K. \tag{1.9}$$

Some B_Ks are found in the last century and mathematicians are still working hard on those unknown Bonnesen-type inequalities ([8], [20], [21]).

Zhang obtains the following Bonneson-style inequalities ([31], [32]):

Proposition 2. *(Zhang) For a convex domain K in R^n, of surface area A and volume V, we have*

$$\left(\frac{A}{n\omega_n}\right)^{\frac{n}{n-1}} - \frac{V}{\omega_n} \geq \left[\left(\frac{V}{\omega_n}\right)^{\frac{1}{n}} - r\right]^n; \tag{1.10}$$

$$\left(\frac{W}{2}\right)^{\frac{n}{n-1}} - \left(\frac{V}{\omega_n}\right)^{\frac{1}{n-1}} \geq \left(\frac{V}{\omega_n}\right)^{\frac{n}{n-1}} \left[\left(\frac{V}{\omega_n}\right)^{-\frac{1}{n}} - R^{-1}\right]^n, \tag{1.11}$$

where r, R, and W are the in-radius, circum-radius, and the mean width of K, respectively.

Bonnesen proved many inequalities of the form (1.10) for two-dimensional cases (cf. [20], [21]) but he was unable to obtain direct generalizations in higher dimensions. This is done very late. The Bonnesen-style inequality (1.10) is derived first by Hadwiger, then the equivalent forms by Dinghas ([13], [14]) and by G. Zhang ([31], [32]) for a convex domain. By evaluating the containment measure, Zhang obtains Bonneson-style inequalities (1.10) and (1.11). The inequality (1.11) is new. For $n = 2$ inequality (1.10) reduce to Bonnesen's original inequality

$$L^2 - 4\pi A \geq (P - 2\pi r)^2. \tag{1.12}$$

The higher dimensional Bonessen-type inequalities are one of works of the corresponding authors [40].

When mathematicians are mainly interested in and focus on the lower bound B_K of the isoperimetric deficit, there is another important and interesting question: is there an invariant U_K of geometric significance such that

$$\Delta_n(K) = A^n - n^n\omega_n V^{n-1} \leq U_K? \tag{1.13}$$

Of course we expect that the upper bound U_K vanishes when K is a disc.

The study of the isoperimetric deficit has a long story, and is still one of the main focuses in geometry and analysis. Unfortunately we are not aware of any general upper bound U_K up today except for few special cases ([7], [22], [24], [31], [36], [37], [34], [42]).

Assume that the boundary ∂K of the convex set K in R^2 has a continuous radius of curvature ρ. Let ρ_m and ρ_M be the smallest and the greatest values, respectively, of ρ. In 1933, Bottema (see [7], [24]) found an upper isoperimetric deficit limit of K, that is,

$$\Delta_2(K) = P^2 - 4\pi A \leq \pi^2(\rho_M - \rho_m)^2, \tag{1.14}$$

where P and A are, respectively, the length of ∂K and the area of K. The equality sign holds if and only if $\rho_M = \rho_m$, that is, K is a disc.

In 1955, Pleijel (see [22], [24]) obtained that

$$\Delta_2(K) = P^2 - 4\pi A \leq \pi(4 - \pi)(\rho_M - \rho_m)^2, \tag{1.15}$$

which is an improvement of Bottema's isoperimetric upper limit.

An analogue of the Bottema's isoperimetric deficit upper limit for convex domain in a plane X_ϵ^2 of constant curvature is obtained by Li and Zhou [17]:

Proposition 3. *(Li, Zhou) Let $\mathbb{X}_\epsilon$ be a complete and simply connected two dimensional surface of constant curvature ϵ, K a convex domain in $\mathbb{X}_\epsilon$. We assume that ∂K is at least C^2 smooth and the geodesic curvature of ∂K satisfies $\kappa_g > \sqrt{-\epsilon}$ when $\epsilon < 0$. If ∂K has a continuous curvature radius ρ, then*

$$\Delta(K) = L^2 - 4\pi A + \epsilon A^2 \leq \left(2\pi - \frac{\epsilon}{2}A\right)^2 \left(\mathrm{tn}_\epsilon \frac{\rho_M}{2} - \mathrm{tn}_\epsilon \frac{\rho_m}{2}\right)^2, \tag{1.16}$$

where ρ_{M} and ρ_{m} are, respectively, the maximum and minimum of ρ. The equality sign holds if and only if K is a geodesic disk.

Here

$$\mathrm{sn}_\epsilon(r) = \begin{cases} \frac{1}{\sqrt{\epsilon}} \sin(\sqrt{\epsilon} r); & \epsilon > 0, \\ r; & \epsilon = 0, \\ \frac{1}{\sqrt{-\epsilon}} \sinh(\sqrt{-\epsilon} r); & \epsilon < 0, \end{cases}$$

and

$$\mathrm{cn}_\epsilon(r) = \begin{cases} \cos(\sqrt{\epsilon} r); & \epsilon > 0, \\ 1; & \epsilon = 0, \\ \cosh(\sqrt{-\epsilon} r); & \epsilon < 0, \end{cases}$$

$$\mathrm{tn}_\epsilon = \mathrm{sn}_\epsilon / \mathrm{cn}_\epsilon, \quad \mathrm{ct}_\epsilon = 1/\mathrm{tn}_\epsilon.$$

We are not aware of any analogue of the Battema's isoperimetric deficit upper limit for convex body in space and even for a general convex domain in R^2 until recent works of Zhou, Ma, Yue and others ([1], [4], [8], [10], [16], [17], [22], [23], [25], [32]), ([38], [36], [42]).

In this paper, we investigate the geometric inequality for a convex body in the Euclidean space R^n. We obtain some isoperimetric deficit upper limits for a convex body. The isoperimetric deficit upper limits obtained are intrinsic invariants involving the surface area, volume, width, diameter of the convex body K. The special planar case is more fundamental than the Bottema's results.

2 The Isoperimetric Deficit Upper Limit of the Convex Body

To obtain our main theorem we need the following results ([2], [37], [42]):

Lemma 1. *(Steinhagen) Let K be a convex body of width W in the Euclidean space R^n and r be the maximum inscribed radius of K. Then*

$$r \geq \alpha_n W, \tag{2.1}$$

where α_n is a constant

$$\alpha_n = \begin{cases} \frac{\sqrt{n+2}}{2(n+1)}, & n \text{ is even}, \\ \frac{1}{2\sqrt{n}}, & n \text{ is odd}. \end{cases} \tag{2.2}$$

Lemma 2. *(Zhou, Ma, Li) Let K be a convex body of surface area A in the Euclidean space R^n, and V the volume enclosed by K. Let D and r be, respectively, the diameter and the maximum inscribed radius of K. Then we have*

$$r \leq \frac{nV}{A} \leq \sqrt[n]{\frac{V}{\omega_n}} \leq \sqrt[n-1]{\frac{A}{n\omega_n}} \leq \frac{D}{2}. \tag{2.3}$$

The equalities hold when K is a ball.

Now we are in the position to produce our main results.

Theorem 1. *Let K be a convex body of surface area A in the Euclidean space R^n, and V the volume enclosed by K. Let D and W be, respectively, the diameter and the width of K. Then we have*

$$\begin{aligned} \Delta_n(K) = A^n - n^n\omega_n V^{n-1} &\leq n^n\omega_n^n \left[\left(\tfrac{D}{2}\right)^{n(n-1)} - (\alpha_n W)^{n(n-1)}\right]. \\ \Delta_n(K) = A^n - n^n\omega_n V^{n-1} &\leq \tfrac{A^n\omega_n}{V} \left[\left(\tfrac{D}{2}\right)^{n} - (\alpha_n W)^{n}\right]. \\ \Delta_n(K) = A^n - n^n\omega_n V^{n-1} &\leq n\omega_n A^{n-1} \left[\left(\tfrac{D}{2}\right)^{(n-1)} - (\alpha_n W)^{(n-1)}\right]. \end{aligned} \tag{2.4}$$

Proof: Via inequalities of Lemma 1 and Lemma 2 we have

$$\alpha_n W \leq \frac{nV}{A} \leq \sqrt[n]{\frac{V}{\omega_n}} \leq \sqrt[n-1]{\frac{A}{n\omega_n}} \leq \frac{D}{2}. \tag{2.5}$$

Then by inequalities

$$\alpha_n W \leq \sqrt[n]{\frac{V}{\omega_n}} \leq \sqrt[n-1]{\frac{A}{n\omega_n}} \leq \frac{D}{2}$$

we have

$$A^n - n^n\omega_n V^{n-1} \leq n^n\omega_n^n \left[\left(\frac{D}{2}\right)^{n(n-1)} - (\alpha_n W)^{n(n-1)}\right].$$

Via inequalities

$$\alpha_n W \leq \frac{nV}{A} \leq \sqrt[n]{\frac{V}{\omega_n}} \leq \frac{D}{2},$$

we obtain

$$A^n - n^n\omega_n V^{n-1} \leq \frac{A^n\omega_n}{V} \left[\left(\frac{D}{2}\right)^{n} - (\alpha_n W)^{n}\right].$$

Finally from

$$\alpha_n W \leq \frac{nV}{A} \leq \sqrt[n-1]{\frac{A}{n\omega_n}} \leq \frac{D}{2}$$

we have

$$A^n - n^n\omega_n V^{n-1} \leq n\omega_n A^{n-1}\left[\left(\frac{D}{2}\right)^{n-1} - (\alpha_n W)^{n-1}\right],$$

and then we complete the proof of the theorem.

When $n = 2$ we immediately have the following corollary of Theorem 1.

Theorem 2. *Let K be a convex domain of area A and circum length P in the Euclidean plane R^2. Let D and W be, respectively, the diameter and width of K. Then we have*

$$\begin{aligned} \Delta_2(K) = L^2 - 4\pi A &\leq \tfrac{\pi^2}{9}(9D^2 - 4W^2), \\ \Delta_2(K) = L^2 - 4\pi A &\leq \tfrac{\pi P^2}{36A}(9D^2 - 4W^2), \\ \Delta_2(K) = L^2 - 4\pi A &\leq \tfrac{\pi P}{3}(3D - 2W). \end{aligned} \tag{2.6}$$

Compare with the results of Bottema and Pleijel, one may see our results are more fundamental since our convex domain does not need to be the oval one.

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